

Evolution by mean curvature flow of Lagrangian spherical surfaces in complex Euclidean plane

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Abstract

We describe the evolution under the mean curvature flow of Lagrangian spherical surfaces in the complex Euclidean plane \mathbb{C}^2 . In particular, for embedded surfaces, we answer the Question 4.7 addressed in [12] about finding out a condition on a starting Lagrangian torus in \mathbb{C}^2 such that the corresponding mean curvature flow becomes extinct at finite time and converges after rescaling to the Clifford torus. On the other hand, we also provide examples of Lagrangian surfaces with self-intersection which develop Type II singularities under the mean curvature flow.

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1 Introduction

Let $F_0 : M^n \rightarrow \mathbb{R}^m$ be an immersion of a compact manifold of dimension $n \geq 2$ into Euclidean space. The mean curvature flow with initial condition F_0 is a smooth family of immersions $F : M \times [0, T) \rightarrow \mathbb{R}^m$ satisfying

$$(MCF) \quad \frac{\partial}{\partial t} F(p, t) = H(p, t), \quad p \in M, \quad t \geq 0; \quad F(\cdot, 0) = F_0,$$

where $H(p, t)$ is the mean curvature vector of the submanifold $M_t = F(M, t)$ at p . It is well-known that (MCF) is a quasilinear parabolic system that is invariant under reparametrizations of M and isometries of the ambient space and short-time existence and uniqueness is guaranteed, being $T < \infty$ the maximal time of existence.

The first classical works in this topic studied the evolution of hypersurfaces by their mean curvature. We emphasize Huisken's paper [7] on the flow of convex surfaces into spheres, proving that if the initial hypersurface is uniformly convex, then the mean curvature flow converges to a round point in finite time. That is, the shape of M_t approaches the shape of a sphere very rapidly and no singularities will occur before the hypersurfaces M_t shrink down to a single point after a finite time. Recently, mean curvature flow of higher codimension submanifolds has also received interest by many authors who have paid attention mainly to graphical submanifolds and symplectic or Lagrangian submanifolds. We

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recall that Huisken's monotonicity formula [8], relating the formation of singularities to self-shrinking solutions of the mean curvature flow, also applies in any codimension. Concretely, the so-called Type I singularities forming in Euclidean space look like self-similar contracting solutions after an appropriate rescaling procedure. According to [16], this type of singularities usually occur when there exists some kind of pinching of the second fundamental form. Andrews and Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying suitable pinching condition and showed that such submanifolds contract to round points.

In this paper we are interested in the class of Lagrangian immersions in complex Euclidean space $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, which is a preserved class under the mean curvature flow. We notice that there do not exist Lagrangian self-shrinking spheres (see [3] or [16] and references therein) and, in addition, Smoczyk showed that the class of smooth closed Lagrangian immersions in \mathbb{C}^n is not δ -pinchable for any δ (see Section 4.1 in [16]). The authors do not know any available result regarding convergence of compact Lagrangians in \mathbb{C}^n . In fact, the following problem was posed by André Neves (Question 7.4 in [12]) as a Lagrangian analogue of Huisken's classical result [7] for the mean curvature flow of convex spheres:

Find a condition on a Lagrangian torus in \mathbb{C}^2 , which implies that the Lagrangian mean curvature flow $(M_t)_{0 \leq t < T}$ will become extinct at time T and, after rescale, M_t converges to the Clifford torus.

Our contribution to this problem is the following main result.

Theorem A. *Let M_0 be an embedded Lagrangian compact surface of \mathbb{C}^2 which is contained in some hypersphere $\mathbb{S}^3(R_0)$ of radius $R_0 > 0$. Then the mean curvature flow (MCF) with initial condition M_0 has a unique solution defined on a maximal interval $[0, T)$, $T \leq R_0^2/4$. In addition:*

- (a) *If M_0 divides $\mathbb{S}^3(R_0)$ in two connected components of equal volume, then $T = R_0^2/4$, the limit M_T of the evolving surfaces M_t when $t \rightarrow T$ is a point and, after rescaling the flow by multiplication by $1/\sqrt{R_0^2 - 4t}$, the limit is a Clifford torus in $\mathbb{S}^3 := \mathbb{S}^3(1) \subset \mathbb{C}^2$.*
- (b) *If M_0 divides $\mathbb{S}^3(R_0)$ in two connected components of different volumes (being $2\pi A_0 R_0^3$ the lowest volume), then $T = A_0 R_0^2 / 2\pi$, the limit M_T of the evolving surfaces M_t when $t \rightarrow T$ is a circle of radius $R_0 \sqrt{1 - 2A_0/\pi}$ and, after rescaling the flow by multiplication by $\sqrt{A_0(R_0^2 - 4t)/(A_0 R_0^2 - 2\pi t)}$, the limit is a cylinder in $\mathbb{R}^3 \subset \mathbb{C}^2$.*

In Proposition 2.1 we show that any compact Lagrangian surface of complex Euclidean plane contained in some hypersphere must be the preimage of a spherical closed curve by the corresponding Hopf fibration, providing in general an immersed torus that was called a Hopf torus by Pinkall in [14]. As we shall see in the proof of Theorem A, A_0 coincides with the area enclosed by the spherical curve $\pi(M_0/R_0)$, projection of $(1/R_0)M_0 \subset \mathbb{S}^3$ on $\mathbb{S}^2(1/2)$ by the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$. The isometry type of the torus M_0 depends not only on the length of the spherical curve $\pi(M_0/R_0)$ but also on the enclosed

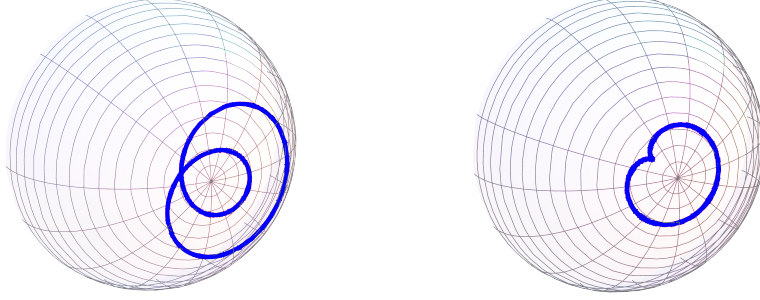


Figure 1: Example of spherical curve of the statement of Theorem B at the initial time and at the time of the singularity.

area A_0 . It was proved in [14] that a Hopf torus M_0 is a critical point of the Willmore functional if and only if its corresponding spherical curve is an elastic curve.

Part (a) of Theorem A is our answer to the Neves question quoted before for a Lagrangian embedded torus M_0 . We point out that the hypotheses on M_0 established in Theorem A are preserved by the mean curvature flow (see Lemma 3.2). Thinking of the shape of the closed spherical elastic curves, M_0 could be a Willmore torus. We remark that the first two authors provided in [3] four rigidity results for the Clifford torus in the class of compact self-shrinkers for the Lagrangian mean curvature flow.

All the singularities appearing in Theorem A are of Type I (see Remark 3.6). We can get Type II singularities by evolution of some not embedded immersed tori. For this aim, we will consider certain spherical convex closed curves with one self-intersection (as for example the one depicted in Figure 1). In fact, we shall prove the following theorem.

Theorem B. *Let M_0 be an immersed Lagrangian compact surface of \mathbb{C}^2 which is contained in some hypersphere $\mathbb{S}^3(R_0)$ of radius $R_0 > 0$ and $\pi_{R_0} : \mathbb{S}^3(R_0) \rightarrow \mathbb{S}^2(R_0/2)$ the Hopf fibration. Let $\pi_{R_0}(M_0)$ be a convex closed curve with one self-intersection such that the areas $A_1 R_0^2$ and $A_2 R_0^2$, enclosed by the inner and outer loops of the curve respectively, satisfy $3A_1 < A_2 < \pi/2$. Then the mean curvature flow (MCF) with initial condition M_0 has a unique solution defined on a maximal interval $[0, T)$, with $T \leq A_1 R_0^2 / \pi$. In addition, the limit M_T of the evolving surfaces M_t when $t \rightarrow T$ is contained in a sphere of radius $R(T) < R_0$, $\pi_{R(T)}(M_T)$ is a curve with cusp in $\mathbb{S}^2(R(T)/2)$ and, after an appropriate rescaling, there is a sequence of times $t_n \rightarrow T$ such that the limit of M_{t_n} when $t_n \rightarrow T$ is a cylinder $\mathbb{R} \times \mathcal{G}$ in $\mathbb{R}^3 \subset \mathbb{C}^2$, where \mathcal{G} is the Grim-Reaper curve.*

1.1 The ideas behind the main results

We now expose some ideas showing that the evolutions considered in Theorems A and B are natural in some geometric sense since they (and some other studied in [6], [10], and [11]) can be regarded as evolutions related with geometric flows of planar and spherical curves.

Let $\alpha_0 : I_1 \rightarrow \mathbb{C}^*$ be a planar regular curve and $\gamma_0 : I_2 \rightarrow \mathbb{S}^2(1/2)$ be a regular spherical curve in \mathbb{C}^2 , where I_1 and I_2 are intervals in \mathbb{R} . Let

$$F_0 : I_1 \times I_2 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{C}^2, \quad F_0(x, y) = \alpha_0(x)\tilde{\gamma}_0(y),$$

with $\tilde{\gamma}_0 : I_2 \rightarrow \mathbb{S}^3 \subset \mathbb{C}^2$ a horizontal lift of γ_0 via the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$. We denote by $\langle \cdot, \cdot \rangle$ and J the Euclidean metric and the complex structure in \mathbb{C}^2 and consider simultaneously a one-parameter family of planar curves

$$\alpha = \alpha(x, t) \in \mathbb{C}^*, \quad t \geq 0, \quad \text{with } \alpha(x, 0) = \alpha_0(x), \quad x \in I_1,$$

and a one-parameter family of spherical curves

$$\gamma = \gamma(y, t) \in \mathbb{S}^2(1/2), \quad t \geq 0, \quad \text{with } \gamma(y, 0) = \gamma_0(y), \quad y \in I_2,$$

and define (see [15]) the Lagrangians

$$(1) \quad F = F(x, y, t) = \alpha(x, t)\tilde{\gamma}(y, t), \quad t \geq 0, \quad (x, y) \in I_1 \times I_2 \subseteq \mathbb{R}^2,$$

where $\tilde{\gamma} = \tilde{\gamma}(y, t) \in \mathbb{S}^3$ is a horizontal lift of $\gamma = \gamma(y, t)$ via the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$.

It is clear that $F(x, y, 0) = F_0(x, y)$. Our goal is to analyse the possible evolutions of α and γ in order to F be a solution of (MCF). Using [15] and the Lagrangian character of each $F_t := F(\cdot, \cdot, t)$, $t \geq 0$, it is not difficult to get that F is a solution of (MCF) if and only if the following two equations (corresponding to the normal directions $J(F_t)_x$ and $J(F_t)_y$) are satisfied:

$$(2) \quad \left\langle \frac{\partial \alpha}{\partial t}, i\alpha' \right\rangle + \langle \alpha, \alpha' \rangle \left\langle \frac{\partial \tilde{\gamma}}{\partial t}, J\tilde{\gamma} \right\rangle = |\alpha'| \kappa_\alpha + \frac{\langle \alpha', i\alpha \rangle}{|\alpha|^2}$$

and

$$(3) \quad |\alpha|^2 \left\langle \frac{\partial \tilde{\gamma}}{\partial t}, J\dot{\tilde{\gamma}} \right\rangle = |\dot{\tilde{\gamma}}| \kappa_{\tilde{\gamma}}.$$

Here $'$ (resp. $\dot{\cdot}$) means derivative respect to x (resp. y) and κ will always denote curvature of the corresponding curve along the paper. Looking at (3) we distinguish two complementary cases:

Case (i): there is no (normal) evolution for $\tilde{\gamma} = \tilde{\gamma}(y, t)$ (and hence for $\gamma = \gamma(y, t)$) and so $\tilde{\gamma}$ (and γ) must be a static geodesic, say

$$\tilde{\gamma}(y, t) = (\cos y, \sin y), \quad \forall t \geq 0.$$

Then equation (2) can be easily rewritten as

$$\left(\frac{\partial \alpha}{\partial t} \right)^\perp = \vec{\kappa}_\alpha - \frac{\alpha^\perp}{|\alpha|^2},$$

where $\vec{\kappa}_\alpha$ is the curvature vector of α and α^\perp denotes the normal component of α . Putting this information in (1) we arrive at the evolution studied in [6], [10] and [11].

Case (ii): necessarily $|\alpha|$ only depends on time variable t , say $R(t) := |\alpha|$. This means that the evolution of $\alpha = \alpha(x, t)$ consists of concentric circles centered at the origin and, up to reparametrizations, it can be given by $\alpha(x, t) = R(t)e^{ix}$. Now (2) translates into a simple o.d.e. for $R(t)$, concretely $-R dR/dt = 2$, whose general solution is $R(t) = \sqrt{R(0)^2 - 4t}$. Putting this in (1), we get that in this case F can be written as

$$(4) \quad F(x, y, t) = \sqrt{R_0^2 - 4t} e^{ix} \tilde{\gamma}(y, t), \quad 0 \leq t < \frac{R_0^2}{4},$$

with $R_0 = R(0)$ and where $\tilde{\gamma}(y, t)$ satisfy now the equation, coming from (3), given by

$$(5) \quad \left\langle \frac{\partial \tilde{\gamma}}{\partial t}, \frac{J \dot{\tilde{\gamma}}}{|\dot{\tilde{\gamma}}|} \right\rangle = \frac{\kappa_{\tilde{\gamma}}}{R_0^2 - 4t}.$$

Using that the Hopf fibration π is a Riemannian submersion, we rewrite (5) as

$$(6) \quad \left\langle \frac{\partial \gamma}{\partial t}, \frac{\gamma \times \dot{\gamma}}{|\dot{\gamma}|} \right\rangle = \frac{2\kappa_\gamma}{R_0^2 - 4t},$$

where \times denotes the cross product in \mathbb{R}^3 . We will check in Section 3 that (6) is essentially the curve shortening flow in $\mathbb{S}^2(1/2)$. The relation between this flow and the corresponding flow (4) of the initial Lagrangian surface will lead to different situations and their study in depth allows us to prove Theorems A and B in Sections 3 and 4 respectively.

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2 Preliminaries

2.1 About Riemannian submersions

In this section we will recall well-known facts on Riemannian submersions and, at the same time, we will introduce most of our notation.

Let $\pi : \overline{M} \longrightarrow \widehat{M}$ be a Riemannian submersion. A vector $v \in T\overline{M}$ is horizontal if it is orthogonal to the fibres ($v \in \ker(d\pi)^\perp$) and vertical if it is tangent to the fibers ($v \in \ker(d\pi)$).

We take $P \subset \widehat{M}$ a Riemannian submanifold with second fundamental form $\widehat{\sigma}$ and mean curvature vector \widehat{H} . Then $M = \pi^{-1}(P) \subset \overline{M}$ is a Riemannian submanifold with second fundamental form $\overline{\sigma}$ and mean curvature vector \overline{H} . Moreover, \overline{M} will be a Riemannian submanifold of \widetilde{M} with second fundamental form $\widetilde{\sigma}$ and mean curvature vector \widetilde{H} . Then $M \subset \widetilde{M}$ is a submanifold of \widetilde{M} with second fundamental form σ and mean curvature vector H (see (7)).

$$(7) \quad \begin{array}{ccccc} M & \hookrightarrow & \overline{M} & \hookrightarrow & \widetilde{M} \\ \pi \downarrow & & \downarrow \pi & & \\ P & \hookrightarrow & \widehat{M} & & \end{array}$$

If X is a vector field tangent to P or \widehat{M} , X^* will denote its horizontal lift tangent to M or \overline{M} , respectively. Given X, Y, Z vector fields tangent to P we get that

$$\pi_*(\overline{\nabla}_{X^*} Y^*) = \widehat{\nabla}_X Y,$$

where $\overline{\nabla}$ and $\widehat{\nabla}$ denote the Riemannian connections of \overline{M} and \widehat{M} , respectively. As a consequence, $\overline{\sigma}(X^*, Y^*) = \widehat{\sigma}(X, Y)^*$ and $\overline{H} = \widehat{H}^* + H_V$, where H_V is the projection to the orthogonal bundle to M of the mean curvature of the fibers of the submersion. Moreover, in this situation we get the following formulas:

$$(8) \quad \begin{aligned} \sigma(X^*, Y^*) &= \overline{\sigma}(X^*, Y^*) + \widetilde{\sigma}(X^*, Y^*) = \widehat{\sigma}(X, Y)^* + \widetilde{\sigma}(X^*, Y^*), \\ H &= \overline{H} + \sum_i \widetilde{\sigma}(e_i^*, e_i^*) + \sum_j \widetilde{\sigma}(f_j, f_j) = \widehat{H}^* + H_V + \sum_i \widetilde{\sigma}(e_i^*, e_i^*) + \sum_j \widetilde{\sigma}(f_j, f_j), \end{aligned}$$

where $\{e_i\}$ is an arbitrary orthonormal frame of TP and $\{f_j\}$ is an arbitrary orthonormal frame of the vertical distribution on \overline{M} .

2.2 Variation of the mean curvature vector by an homothety of the ambient space

Let \overline{M} and \overline{M}_R be Riemannian manifolds with Riemannian metrics \overline{g} and \overline{g}_R respectively. Given a diffeomorphism $\phi_R : \overline{M} \rightarrow \overline{M}_R$ such that $\phi^* \overline{g}_R = R^2 \overline{g}$, $R > 0$ (i.e. the metrics \overline{g} and \overline{g}_R are homothetic), the Koszul formula tells us that the Levi-Civita connection of the metric does not change by homotheties, i.e. $\overline{\nabla}_{\phi_* A} \phi_* B = \overline{\nabla}_A B$, where A and B are tangent vector fields in \overline{M} . Moreover, the relation between the second fundamental forms of an immersion in \overline{M} and \overline{M}_R respectively is given by

$$\overline{\sigma}_R(\phi_* A, \phi_* B) = (\overline{\nabla}_{\phi_* A} \phi_* B)^\perp = \phi_*(\overline{\nabla}_A B)^\perp = \phi_*(\overline{\sigma}(A, B)).$$

Finally, we get the following relation for the respective mean curvature vectors:

$$(9) \quad \overline{H}_R = \frac{1}{R^2} \sum_i \overline{\sigma}_R(\phi_*(e_i), \phi_*(e_i)) = \frac{1}{R^2} \sum_i \phi_*(\overline{\sigma}(e_i, e_i)) = \frac{1}{R^2} \phi_* \overline{H}$$

where $\{e_i\}$ is an arbitrary orthonormal frame of $T\overline{M}$ with respect to \overline{g} (and so $\{e_i/R\}$ is an orthonormal frame of $T\overline{M}_R$ with respect to \overline{g}_R).

2.3 Concretion in the case of the Hopf fibration

In this section we consider the Hopf fibration as a particular case of Riemannian submersion. Let $\overline{M} = \mathbb{S}^3(R)$ be the 3-sphere of radius R in $\mathbb{C}^2 \equiv \mathbb{R}^4$, $\widehat{M} = \mathbb{S}^2(R/2)$ the 2-sphere of radius $R/2$ and $\pi_R : \mathbb{S}^3(R) \rightarrow \mathbb{S}^2(R/2)$ the Hopf fibration given by

$$\pi_R(z, w) = \frac{1}{2R} (2z\overline{w}, |z|^2 - |w|^2), \quad (z, w) \in \mathbb{S}^3(R) \subset \mathbb{C}^2.$$

When $R = 1$, we will omit the subindex R .

In this situation, we take P_R as a closed curve in $\mathbb{S}^2(R/2)$ which we will parametrize by $\gamma_R(v)$, $v \in [0, 2\pi]$, where $\gamma_R : \mathbb{S}^1 \equiv [0, 2\pi]/\sim \rightarrow \mathbb{S}^2(R/2)$, and define $M_R \subset \mathbb{S}^3(R)$ the Riemannian surface $\pi_R^{-1}(P_R)$ given by its position vector F_R in $\mathbb{S}^3(R)$; we remark that $F_R : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3(R)$. Therefore, the diagram (7) translates now into

$$\begin{array}{ccccc} M_R \equiv \mathbb{S}^1 \times \mathbb{S}^1 & \xrightarrow{F_R} & \mathbb{S}^3(R) & \hookrightarrow & \mathbb{C}^2 \\ & & \downarrow \pi_R & & \\ P_R \equiv \mathbb{S}^1 & \xrightarrow{\gamma_R} & \mathbb{S}^2(R/2) & & \end{array}$$

We consider the homothetic diffeomorphism $\phi : \mathbb{S}^3 \subset \mathbb{C}^2 \rightarrow \mathbb{S}^3(R) \subset \mathbb{C}^2$ given by $\phi(x) = Rx$, which satisfies that $\phi_*A = RA$ for any vector A tangent to \mathbb{S}^3 . We get the following relation between the second fundamental forms $\tilde{\sigma}_R$ of $\mathbb{S}^3(R)$ in \mathbb{C}^2 and $\tilde{\sigma}$ of \mathbb{S}^3 in \mathbb{C}^2 :

$$\tilde{\sigma}_R(A, B) = -\frac{1}{R}\tilde{g}_R(A, B)\frac{F_R}{R} = -\frac{1}{R}\tilde{g}(A, B)F = -\frac{1}{R}\tilde{\sigma}(A, B),$$

for any tangent vectors A and B . Due to the fact that the fibres of the Hopf fibrations π and π_R are geodesics we know that $H_V = 0 = H_{RV}$, and from (9) we also get that

$$\overline{H}_R = \frac{1}{R^2}\phi_*\overline{H} = \frac{1}{R}\overline{H}.$$

In the same way, using now the diffeomorphism $\phi : \mathbb{S}^2(1/2) \rightarrow \mathbb{S}^2(R/2)$, γ_R can be written as $\gamma_R = \phi \circ \gamma := R\gamma$, with $\gamma : \mathbb{S}^1 \equiv [0, 2\pi]/\sim \rightarrow \mathbb{S}^2(1/2)$. If we denote the unit tangent vector of γ in \mathbb{S}^2 by $e_1 := \frac{\gamma'(u)}{|\gamma'(u)|}$, then

$$e_{R1} := \frac{\gamma'_R(u)}{|\gamma'_R(u)|} = \frac{\phi_*e_1}{|\phi_*e_1|} = \frac{R e_1}{|R e_1|} = \frac{\gamma'(u)}{|\gamma'(u)|} = e_1.$$

Finally, taking into account (8) and (9), we deduce:

$$(10) \quad H_R = \overline{H}_R + \tilde{\sigma}_R(e_1^*, e_1^*) + \tilde{\sigma}_R(JF, JF) = \frac{1}{R}\overline{H} - \frac{2}{R}F = \frac{1}{R}\widehat{H}^* - \frac{2}{R}F,$$

where \widehat{H}^* is the horizontal lift by π of the mean curvature $\widehat{H} \equiv \vec{\kappa}_\gamma$ (i.e. the curvature vector) of γ in $\mathbb{S}^2(1/2)$.

2.4 Spherical Lagrangian submanifolds

In the complex Euclidean plane \mathbb{C}^2 we consider the bilinear Hermitian product defined by

$$(z, w) = z_1\bar{w}_1 + z_2\bar{w}_2, \quad z, w \in \mathbb{C}^2.$$

Then $\langle \cdot, \cdot \rangle = \text{Re}(\cdot, \cdot)$ is the Euclidean metric on \mathbb{C}^2 and $\omega = -\text{Im}(\cdot, \cdot)$ is the Kaehler two-form given by $\omega(\cdot, \cdot) = \langle J\cdot, \cdot \rangle$, where J is the complex structure on \mathbb{C}^2 .

Let $F : M \rightarrow \mathbb{C}^2$ be an isometric immersion of a surface M into \mathbb{C}^2 . F is said to be Lagrangian if $F^*\omega = 0$. This is equivalent to the orthogonal decomposition $T\mathbb{C}^2 = F_*TM \oplus JF_*TM$, where TM is the tangent bundle of M .

Proposition 2.1. *Let M be any compact Lagrangian surface of \mathbb{C}^2 contained in some hypersphere $\mathbb{S}^3(R)$, $R > 0$. Then M must be the preimage of a closed curve in $\mathbb{S}^2(R/2)$ by the Hopf fibration $\pi_R : \mathbb{S}^3(R) \rightarrow \mathbb{S}^2(R/2)$.*

Proof. Let N be the unit vector normal to $\mathbb{S}^3(R)$ in \mathbb{C}^2 . Then JN is a vector field on $\mathbb{S}^3(R)$ whose integral curves are the fibres of the Hopf fibration π_R . Since M is Lagrangian, the restriction of JN to M is a tangent vector field on M and its integral curves are contained in M . In this way, the restriction of π_R to M is a Riemannian submersion on its image $\pi_R(M) =: C$ with the same fibres that the Hopf fibration. That is, $M = \pi_R^{-1}(C)$ for some closed curve $C \subset \mathbb{S}^2(R/2)$. \square

Remark 2.2. If F_R denotes the immersion of M into $\mathbb{S}^3(R) \subset \mathbb{C}^2$, then Proposition 2.1 tells us that F_R can be regarded as $F_R : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3(R)$ and there exists a curve $\gamma_R : \mathbb{S}^1 \rightarrow \mathbb{S}^2(R/2)$ such that $(\pi_R \circ F_R)(u, v) = \gamma_R(v)$ and $F_R(\mathbb{S}^1 \times \{v_0\})$ is a fibre of the Hopf fibration for every $v_0 \in \mathbb{S}^1$.

3 Proof of Theorem A

Let M_t be a one-parameter family of Lagrangian surfaces of \mathbb{C}^2 contained in the spheres $\mathbb{S}^3(R(t))$ of radius $R(t) > 0$. Using Proposition 2.1 and Remark 2.2, this family can be parametrized in the following way:

$$F_{R(t)}(u, v, t) = R(t)F(u, v, t),$$

where $F(\cdot, \cdot, t) : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ is a family of Lagrangian immersions of a torus in \mathbb{C}^2 contained in the unit hypersphere, and there exists a family of curves $\gamma(\cdot, t) : \mathbb{S}^1 \rightarrow \mathbb{S}^2(1/2)$ such that $(\pi_{R(t)} \circ F_{R(t)})(u, v, t) = R(t)\gamma(v, t)$, which is equivalent to

$$(11) \quad (\pi \circ F)(u, v, t) = \gamma(v, t).$$

We study now when this family $F_{R(t)}$ satisfies the mean curvature flow equation (MCF). The left side of (MCF) is obviously

$$(12) \quad \frac{\partial F_R}{\partial t}(u, v, t) = R'(t) F(u, v, t) + R(t) \frac{\partial F}{\partial t}(u, v, t).$$

To compute the right side of (MCF), we will use (10) at each time t . This would be right only if the property of being contained in a sphere is preserved along the flow. But this conservation is a consequence of the fact that we will find a solution assuming this property. Therefore, using (10) and (12), the evolution equation $\partial F_R / \partial t = H_R$ becomes

$$R' F + R \frac{\partial F}{\partial t} = \frac{1}{R} \widehat{H}^* - \frac{2}{R} F.$$

Since $|F| = 1$, necessarily $\frac{\partial F}{\partial t}$ is orthogonal to F , and so the above equation separates in two coupled ones:

$$\begin{cases} R' = -\frac{2}{R} \\ R \frac{\partial F}{\partial t} = \frac{1}{R} \widehat{H}^* \end{cases}$$

Putting $R(0) = R_0$, the solution of the first equation is $R^2(t) = R_0^2 - 4t$. Plugging this solution in the second one, we obtain that

$$\frac{\partial F}{\partial t} = \frac{1}{R_0^2 - 4t} \hat{H}^*.$$

Using (11) and recalling that $\hat{H} = \vec{\kappa}_\gamma$, the composition with π_* of the above equation implies that

$$\frac{\partial \gamma}{\partial t} = \frac{1}{R_0^2 - 4t} \hat{H} = \frac{1}{R_0^2 - 4t} \vec{\kappa}_\gamma.$$

This is not exactly the mean curvature flow for $\gamma(v, t)$; but we consider the change of parameter $t = t(\bar{t})$ given by

$$(13) \quad \bar{t} = \bar{t}(t) = \int_0^t \frac{1}{R_0^2 - 4s} ds = -\frac{1}{4} \ln \frac{R_0^2 - 4t}{R_0^2} = \ln \left(\frac{R_0^2}{R_0^2 - 4t} \right)^{1/4}.$$

In this way, we arrive at

$$(14) \quad \frac{\partial \gamma}{\partial \bar{t}} = \frac{\partial t}{\partial \bar{t}} \frac{1}{R_0^2 - 4t} \hat{H} = \vec{\kappa}_\gamma,$$

which is the mean curvature flow for $\gamma(u, t(\bar{t}))$.

As a summary, we have proved the following result.

Theorem 3.1. *Let F_{R_0} be a Lagrangian immersion of a surface in \mathbb{C}^2 , contained in the hypersphere $\mathbb{S}^3(R_0)$ of radius $R_0 > 0$. Then F_{R_0} evolves under the mean curvature flow following the formula:*

$$(15) \quad F_{R_0}(\cdot, t) = \sqrt{R_0^2 - 4t} F(\cdot, t),$$

where $F(\cdot, t)$ is the preimage by the Hopf fibration $\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2)$ of a curve $\gamma(\cdot, \bar{t}(t))$ satisfying the evolution equation (14), where $\bar{t}(t)$ is the function given in (13).

In order to continue with the proof of Theorem A, we need the following lemma.

Lemma 3.2. *Let γ_0 be a closed simple curve in $\mathbb{S}^2(1/2)$ enclosing a domain with area $A_0 \leq \pi/2$. If $A(\bar{t})$ denotes the area enclosed by a solution $\gamma(\cdot, \bar{t})$ of (14) with initial condition $\gamma(\cdot, 0) = \gamma_0(\cdot)$, then $A(\bar{t}) = \pi/2 - (\pi/2 - A_0) e^{4\bar{t}}$, and the extinction time of $\gamma(\cdot, \bar{t})$ is given by $\tau = \ln \left(\frac{\pi}{\pi - 2A_0} \right)^{1/4} \leq \infty$.*

Proof. It is well known that the rate at which the area $A(\bar{t})$ decrease with time \bar{t} is given by $\partial A / \partial \bar{t} = - \int_\gamma \kappa_\gamma ds$, which implies using the Gauss-Bonnet formula that $A'(\bar{t}) = 4A(\bar{t}) - 2\pi$, taking into account that γ lies in a sphere of radius $1/2$. Solving the former equation, we obtain that $\ln(2\pi - 4A(\bar{t}))^{1/4} = \ln(2\pi - 4A_0)^{1/4} + \bar{t}$, and this proves the statement. \square

Corollary 3.3. *Under the hypothesis of Theorem 3.1 and Lemma 3.2, there are only two possibilities for the evolution under the mean curvature flow of a Lagrangian embedding F_{R_0} of a compact surface in \mathbb{C}^2 :*

- (a) *If $F_{R_0}(\mathbb{S}^1 \times \mathbb{S}^1)$ divides $\mathbb{S}^3(R_0)$ in two connected components of equal volume, then $F_{R_0}(\cdot, t)$ is defined for $t \in [0, R_0^2/4)$, the limit of $F_{R_0}(\cdot, t)$ when $t \rightarrow R_0^2/4$ is the center of $\mathbb{S}^3(R_0)$, and rescaling t by \bar{t} according to (13) and $F_{R_0}(\cdot, t)$ by $\tilde{F}_{R_0}(\cdot, t) = \frac{1}{\sqrt{R_0^2 - 4t}} F_{R_0}(\cdot, t)$, then $\lim_{\bar{t} \rightarrow \infty} \tilde{F}_{R_0}(\cdot, \bar{t})$ is the Clifford torus in \mathbb{S}^3 .*
- (b) *If $F_{R_0}(\mathbb{S}^1 \times \mathbb{S}^1)$ divides $\mathbb{S}^3(R_0)$ in two connected components of different volumes, then $F_{R_0}(\cdot, t)$ is defined for $t \in [0, T)$, $T = A_0 R_0^2 / 2\pi < R_0^2/4$, and the limit of $F_{R_0}(\cdot, t)$ when $t \rightarrow T$ is a circle of radius $\sqrt{R_0^2 - 4T} = R_0 \sqrt{1 - 2\pi/A_0} > 0$, where A_0 is the area enclosed by the curve $\gamma_0(\cdot) = \gamma(\cdot, 0) \subset \mathbb{S}^2(1/2)$.*

Proof. From Theorem 3.1 it follows that the flow $F_{R_0}(\cdot, t)$ given in (15) is defined in $[0, T)$, the intersection of the intervals where $\sqrt{R_0^2 - 4t}$ and $\gamma(\cdot, \bar{t}(t))$ are defined. On the one hand, this implies immediately that $T \leq R_0^2/4$. On the other hand, $\gamma(\cdot, \bar{t})$ is well defined on $[0, \tau)$ (see Lemma 3.2). Using (13), we get that

$$(16) \quad t(\tau) = \frac{R_0^2}{4} (1 - e^{-4\tau}).$$

It is well known for the curve shortening flow in the 2-sphere (see for instance [5] and also [4]) that there are only two possibilities:

- (a) $\tau = \infty$ and $\lim_{\bar{t} \rightarrow \infty} \gamma(\cdot, \bar{t})$ is a geodesic of $\mathbb{S}^2(1/2)$.

This case corresponds to $A_0 = \pi/2$. Then it follows from (16) that $t(\infty) = R_0^2/4$ and so $\lim_{t \rightarrow R_0^2/4} \gamma(\cdot, \bar{t}(t))$ is a geodesic in $\mathbb{S}^2(1/2)$. Thus, the limit of the preimage $F(\cdot, t)$ when $t \rightarrow R_0^2/4$ is the preimage of a geodesic in $\mathbb{S}^2(1/2)$, which is the Clifford torus in \mathbb{S}^3 . Therefore, rescaling t to get \bar{t} and $F_{R_0}(\cdot, t)$ to $\tilde{F}_{R_0}(\cdot, t) = \frac{1}{\sqrt{R_0^2 - 4t}} F_{R_0}(\cdot, t)$, we obtain that

$$\tilde{F}_{R_0}(\cdot, \bar{t}) := \tilde{F}_{R_0}(\cdot, t(\bar{t})) = F(\cdot, t(\bar{t}))$$

and, as we have just deduced, $\lim_{\bar{t} \rightarrow \infty} F(\cdot, t(\bar{t}))$ is the Clifford torus in \mathbb{S}^3 .

- (b) $\tau < \infty$ and $\lim_{\bar{t} \rightarrow \tau} \gamma(\cdot, \bar{t})$ is a point of $\mathbb{S}^2(1/2)$.

This case corresponds to $A_0 < \pi/2$. Using Lemma 3.2 and (16), we have that $T = t(\tau) = A_0 R_0^2 / 2\pi < R_0^2/4$. Moreover, the limit when $t \rightarrow T$ of $\gamma(\cdot, \bar{t}(t))$ is a point of $\mathbb{S}^2(1/2)$, whose preimage is a circle of radius 1 in \mathbb{S}^3 . Thus $\lim_{t \rightarrow T} F_{R_0}(\cdot, t)$ is a circle of radius $\sqrt{R_0^2 - 4T} > 0$ in $\mathbb{S}^3(\sqrt{R_0^2 - 4T})$. \square

In the case (a) of Corollary 3.3 we have used the total space to rescale. However, in the case (b) we will use the base space to rescale. A natural rescaling for the curve γ in $\mathbb{S}^2(1/2)$ shrinking to a point $x \in \mathbb{S}^2(1/2)$ is to consider the 2-sphere in \mathbb{R}^3 and to multiply

$\gamma - x$ by a function of \bar{t} such that the area enclosed by the rescaled curves be constant. According to Lemma 3.2, this rescaling is given by

$$(17) \quad \tilde{\gamma}(\cdot, \bar{t}) - x = \sqrt{\frac{A_0}{\pi/2 - (\pi/2 - A_0)e^{4\bar{t}}}} (\gamma(\cdot, t(\bar{t})) - x).$$

Now a well known result on the curve shortening flow in a surface (see [17]) implies that the limit of the rescaling (17) when $\bar{t} \rightarrow \tau$ (that is, $t \rightarrow T$) is a planar circle centered at x of radius $\sqrt{A_0/\pi}$.

Hence, taking into account the formula given in (13), for the Lagrangian surface $F_{R_0}(\cdot, t)$ we will use the rescaling

$$(18) \quad \tilde{F}_{R_0}(\cdot, t) - q = \sqrt{\frac{A_0}{\pi/2 - (\pi/2 - A_0)\left(\frac{R_0^2}{R_0^2 - 4t}\right)}} (F_{R_0}(\cdot, t) - q)$$

where $q = \sqrt{R_0^2 - 4t} q_1$, being q_1 a point in the limit circle of F when $t \rightarrow T$.

Proposition 3.4. *When $T < R_0^2/4$, the limit of the rescaling (18) when $t \rightarrow T$ is a cylinder passing through $\sqrt{R_0^2 - 4T} q_1$, which is the product of a circle of radius $\sqrt{(R_0^2 - 4T)A_0/\pi}$ and a line.*

Proof. Let us denote

$$\lambda \equiv \lambda(t) := \sqrt{\frac{A_0}{\pi/2 - (\pi/2 - A_0)\left(\frac{R_0^2}{R_0^2 - 4t}\right)}}.$$

We remark that $\lambda \rightarrow \infty$ when $t \rightarrow T = A_0 R_0^2 / 2\pi$ and recall that $R(t) = \sqrt{R_0^2 - 4t}$.

First we check that the rescalings \tilde{F}_{R_0} and $R(t)\tilde{\gamma}$, given by (18) and (17) respectively, are also related by a Hopf fibration

$$\tilde{\pi}_t : \mathbb{S}^3((1 - \lambda)q, \lambda R(t)) \longrightarrow \mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2).$$

For this purpose, we consider the transformations

$$\mu_t : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \quad z \mapsto q + \lambda(z - q)$$

and

$$\nu_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad w \mapsto R(t)x + \lambda(w - R(t)x).$$

Then (18) and (17) can be rewritten as

$$\tilde{F}_{R_0}(\cdot, t) = \mu_t(F_{R_0}(\cdot, t)), \quad R(t)\tilde{\gamma}(\cdot, t) = \nu_t(R(t)\gamma(\cdot, t)),$$

and, in addition, we have that

$$\mu_t(\mathbb{S}^3(R(t))) = \mathbb{S}^3((1 - \lambda)q, \lambda R(t)), \quad \nu_t(\mathbb{S}^2(R(t)/2)) = \mathbb{S}^2((1 - \lambda)R(t)x, \lambda R(t)/2).$$

Accordingly, the Hopf maps $\pi_{R(t)}$ and $\tilde{\pi}_t$ are then related by

$$\tilde{\pi}_t(z) = (\nu_t \circ \pi_{R(t)} \circ \mu_t^{-1})(z) = \pi_{\lambda R(t)}(-(1-\lambda)q + z) + (1-\lambda)R(t)x.$$

As a consequence, $(\tilde{\pi}_t \circ \tilde{F}_{R_0})(\cdot, t) = R(t)\tilde{\gamma}(\cdot, t)$.

We study now the limit when $t \rightarrow T$ (which implies $\lambda \rightarrow \infty$). For this purpose, we define $H_3 = R(T)q_1 + \{q_1\}^\perp \subset \mathbb{C}^2$ and $H_2 = R(T)x + \{x\}^\perp \subset \mathbb{R}^3$. When $t \rightarrow T$, the geodesics $\gamma_{q,v,t}$ of $\mathbb{S}^3((1-\lambda)q, \lambda R(t))$ passing through q and tangent to some unit tangent vector v (necessarily orthogonal to q) go to the lines in H_3 passing through q and tangent to v .

It is known (see, for instance, formula (3.5) in [9]) that the image of a geodesic of $\mathbb{S}^3(R)$ by the Hopf map is a round circle of $\mathbb{S}^2(R/2)$. Then $\tilde{\pi}_t \circ \gamma_{q,v,t}$ is a round circle of $\mathbb{S}^2((1-\lambda)R(t)x, \lambda R(t)/2)$ passing through $R(t)x$ and tangent to $\tilde{\pi}_{t*}v$, which is orthogonal to $x = \tilde{\pi}_t q_1$. When $t \rightarrow T$, the limit of each of these round circles is the line in H_2 passing through $R(T)x$ in the direction of $\lim_{t \rightarrow T} \tilde{\pi}_{t*}v$.

Moreover, for any t , $\ker \tilde{\pi}_{t*} = \text{span}\{Jq_1\}$. Then the limit of $\tilde{\pi}_t$ when $t \rightarrow T$ is a map $\tilde{\pi}_T$ from H_3 to H_2 given by $\tilde{\pi}_T(R(T)q_1 + \alpha Jq_1 + w) = R(T)x + f(w)$, where w is any vector in the subspace W orthogonal to the space generated by q_1 and Jq_1 in \mathbb{C}^2 and f is an isometry between W and the orthogonal subspace to x in \mathbb{R}^3 . In other words, $\tilde{\pi}_T$ is an “orthogonal” projection of H_3 onto H_2 with “fibre in the direction of Jq ”.

Finally, since the limit of $R(t)\tilde{\gamma}(\cdot, t)$ when $t \rightarrow T$ is the circle centered at $R(T)x$ of radius $R(T)\sqrt{A_0/\pi}$, and $R(t)\tilde{\gamma}(\cdot, t) = (\tilde{\pi}_t \circ \tilde{F}_{R_0})(t)$, then the limit of $\tilde{F}_{R_0}(t)$ when $t \rightarrow T$ is the preimage by the limit projection $\tilde{\pi}_T$ of the above circle, which is the cylinder described in the statement of the Proposition. \square

Remark 3.5. We observe that the rescaling (18) is not exactly the standard one given in [7]. Nevertheless, they only differ in the product by a bounded function and consequently they are equivalent.

Remark 3.6. All the singularities appearing in Theorem A are Type I singularities. In fact, following Section 2 and using Theorem 3.1, it is not difficult to check that the second fundamental form σ of the evolution (15) is given by

$$|\sigma|^2 = \frac{4 + \kappa_\gamma^2}{R_0^2 - 4t}, \quad t \in [0, T).$$

In case (a), we have that $T = R_0^2/4$ and we know that κ_γ is bounded by some constant L ; then we get that $(T-t)|\sigma|^2 = 1 + \kappa_\gamma^2/4 \leq 1 + L/4$, which implies the condition of being a Type I singularity.

In case (b), we have that $T = t(\tau) = A_0 R_0^2/2\pi < R_0^2/4$ and we know that γ develops a Type I singularity. So there exists a constant C such that $(\tau - \bar{t})\kappa_\gamma^2 \leq C$. Using that $A_0 < \pi/2$ and (13), we get that

$$(T-t)|\sigma|^2 < 1 + \frac{(T-t)\kappa_\gamma^2}{R_0^2 - 4t} = 1 + \frac{1 - e^{4(\bar{t}-\tau)}}{4}\kappa_\gamma^2.$$

If we define $G(\bar{t}) = (1 - e^{4(\bar{t}-\tau)})/4 - (\tau - \bar{t})$, it is easy to check that $G'(\bar{t}) > 0$ and so $G(\bar{t}) < G(\tau) = 0$. Hence we conclude that $(T - t)|\sigma|^2 < 1 + (\tau - \bar{t})\kappa_\gamma^2 \leq 1 + C$, that shows that the behaviour of $|\sigma|$ in case (b) is determined by the one of $|k_\gamma|$, which corresponds to a Type I singularity.

4 Proof of Theorem B

We start this section introducing first some notation. For our convenience, we will denote by

$$\mu^{-1} : \mathbb{S}^2(1/2) \setminus \{(0, 0, 1/2)\} \rightarrow \mathbb{R}^2$$

the stereographic projection from the North pole to the plane including the equator of $\mathbb{S}^2(1/2)$. Let φ be the function on \mathbb{R}^2 defined by $\mu^*g_{\mathbb{S}^2(1/2)} = \varphi^2(dx^2 + dy^2)$. Concretely, $\varphi(x, y) = 1/(1 + x^2 + y^2)$. It is clear that $\nabla\varphi = -2(x, y)/(1 + x^2 + y^2)^2$. Given a curve α in \mathbb{R}^2 , the curvatures κ_α and $\kappa_{\mu\circ\alpha}$ are related by

$$(19) \quad \kappa_\alpha = \varphi \kappa_{\mu\circ\alpha} + \vec{n}_\alpha(\varphi),$$

where \vec{n}_α is the unit normal vector of α (see formula (3.2) in [13]).

If we assume that $\mu \circ \alpha$ lies in the South hemisphere, as $1/\varphi$, φ and $|\nabla\varphi|$ are bounded on the image by μ^{-1} of the South hemisphere, we can conclude from (19) that there are universal constants a , b , c and d satisfying

$$(20) \quad |\kappa_{\mu\circ\alpha}| \leq a |\kappa_\alpha| + b, \text{ and } |\kappa_\alpha| \leq c |\kappa_{\mu\circ\alpha}| + d.$$

Now we can proceed to the proof of Theorem B. Let M_0 be an immersed Lagrangian surface of \mathbb{C}^2 which is contained in some hypersphere $\mathbb{S}^3(R_0)$ of radius $R_0 > 0$. Using Proposition 2.1 and the hypotheses of Theorem B, M_0 is the preimage by the Hopf fibration of a closed convex curve C_0 in $\mathbb{S}^2(R_0/2)$. Then we can assume that it is contained in the South hemisphere and, following the above notation, we can parametrize it as $C_0 \equiv R_0 \mu \circ \alpha$. Since C_0 is convex, the South pole must be contained in the interior of the domains bounded by its loops and $\mu \circ \alpha$ inherits the same properties. In particular, $\kappa_{\mu\circ\alpha} > 0$. In addition, taking into account the fact that $\mu \circ \varphi$ is a radial function respect to the South pole, we obtain that $\cos(\angle(\vec{n}_{\mu\circ\alpha}, \nabla(\mu \circ \varphi))) > 0$. This last inequality implies $\cos(\angle(\vec{n}_\alpha, \nabla\varphi)) > 0$ because μ is a conformal map. Using the above inequalities in (19), we conclude that $\kappa_\alpha > 0$, that is, α is also a convex curve.

Now we recall some results on the curve shortening flow, that will be used later.

Lemma 4.1 ([13]). *If $\psi : M \rightarrow M'$ is a conformal diffeomorphism between two surfaces and β is a solution to the curve shortening flow in M , then $\psi \circ \beta$ is a solution to the curve shortening flow in M' .*

Lemma 4.2 ([2]). *Let β be a closed convex curve evolving under the curvature shortening flow in \mathbb{R}^2 . If β develops only Type I singularities, then its rescaled curvature $\sqrt{T - t} \kappa_\beta$ converges to the curvature of one of the Abresh-Langer curves. If β develops a Type II singularity, then there is a sequence of times t_n converging to the singularity maximal time T such that the rescaled curve $\beta(\cdot, t_n) = \kappa_\beta(p_n)(\beta(\cdot, t_n) - \beta(p_n, t_n))$, being p_n the point where $\beta(\cdot, t_n)$ reaches its maximal curvature, converges to the Grim Reaper curve.*

If M_0 evolves by the mean curvature flow, Theorem 3.1 ensures us that we can parametrize M_t and C_t by (15) and $R(t)\gamma(\cdot, \bar{t}(t))$ respectively, where $\gamma(\cdot, \bar{t})$ is a family of curves in $\mathbb{S}^2(1/2)$ satisfying (14) and having two loops that we will denote $\gamma_1(\cdot, \bar{t})$ and $\gamma_2(\cdot, \bar{t})$. The initial restrictions on the areas enclosed by $R_0\gamma(\cdot, 0)$ imply that the areas $A_1(\bar{t})$ and $A_2(\bar{t})$, enclosed by the curves $\gamma_1(\cdot, \bar{t})$ and $\gamma_2(\cdot, \bar{t})$ respectively, $3A_1(0) < A_2(0) < \pi/2$.

We apply Gauss-Bonnet Theorem to $\gamma_i(\cdot, \bar{t})$, $i = 1, 2$, and denoting by A'_i the derivatives respect to \bar{t} , we get:

$$(21) \quad A'_1 = 4A_1 + \theta - 2\pi < 4A_1 - \pi, \quad A'_2 = 4A_2 - \theta - 2\pi > 4A_1 - 3\pi$$

where $0 < \theta < \pi$ is the supplementary external angle of the loop $\gamma_1(\cdot, \bar{t})$ at the crossing point.

Let τ_1 be the time such that $A_1(\tau_1) = 0$. It follows from (21) that $A_1(\bar{t}) \leq \pi/4 - (\pi/4 - A_1(0))e^{4\bar{t}}$ and $A_2(\bar{t}) \geq 3\pi/4 - (3\pi/4 - A_2(0))e^{4\bar{t}}$. So if $A_2(0) > 3A_1(0)$ then the area in the smaller loop must disappear before $\frac{1}{4} \ln \left(\frac{\pi}{\pi - 4A_1(0)} \right) \geq \tau_1$, while the area in the larger loop stays positive and bounded from below. Necessarily the curve becomes singular when this happens. Thus the curve $\gamma(\cdot, \bar{t})$ develops a singularity at $\bar{t} = \tau_1$ and, in addition, $T = t(\tau_1) \leq A_1(0)R_0^2/\pi < R_0^2/4$. Therefore $\lim_{t \rightarrow T} F_{R_0}(\cdot, t)$ is the preimage of $R(T)\gamma(\cdot, \bar{t}(T))$ in $\mathbb{S}^3(R(T))$, whose fibres are circles of radius $R(T)$.

From Lemma 4.1, $(\mu^{-1} \circ \gamma)(\cdot, \bar{t})$ is a evolution of convex curves in the plane producing a singularity at time $\bar{t} = \tau_1$. Using Lemma 4.2 and (20), this must be necessarily a Type II singularity and, rescaling it as in the statement of the lemma, $(\mu^{-1} \circ \gamma)(\cdot, \bar{t})$ subconverges to the Grim-Reaper \mathcal{G} . Finally, similar arguments to those used in the proof of Proposition 3.4 conclude that the flow M_t of the Lagrangian surface M_0 subconverges to a cylinder $\mathbb{R} \times \mathcal{G}$ in \mathbb{R}^3 . This finishes the proof of Theorem B.

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